Classification of Closed Surfaces

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Outline

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Classification of Compact Surfaces

Theorem (The Classification of Surfaces)

Every closed, connected surface is homeomorphic to exactly one of:

- *the* **sphere** S^2,
- *a* **connected sum of** n **tori** $T^2 \# \cdots \# T^2 \quad (n \geq 1)$, *or*
- *a* **connected sum of** m **projective planes** $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ $(m \geq 1)$.

The proof follows the same strategy as the classification of **closed prime surfaces**.

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Key Definitions and Facts

Definition

A surface S is **prime** $\iff$ S contains no essential separating simple closed curve.

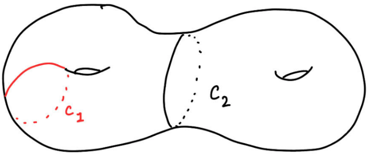


Figure: Essential Separating Curves c_1, c_2

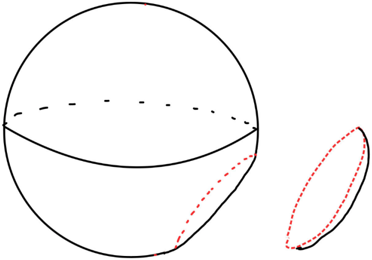
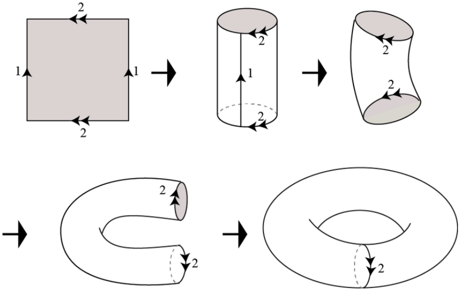


Figure: Inessential separating curve

Key Definitions and Facts

Lemma

Every closed connected surface S admits a polygonal representation.



The proof proceeds by considering a surfaces triangulation (S, K) and it's spanning tree T, and inductively removing vertices from T to construct a polygonal representation of S.

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Step 1: Reducing Adjacent Identified Edges

Lemma

The sphere S^2, the torus $T^2 = S^1 \times S^1$ and the real projective plane $\mathbb{RP}^2$ are the only prime surfaces.

Proof: Let P be a polygonal representation of a prime surface S. Suppose two **adjacent edges** of P are identified like in the picture:

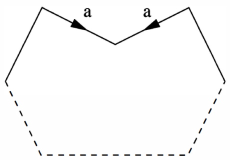

Figure 2.11. *Redundant edges in a polygonal representation of S.*

We may identify them to obtain a polygon with **fewer edges**, so set P to be the polygonal representation of S with the fewest possible number of edges.

Special case: Suppose P has two adjacent edges pointing in the same direction, with *no other edges*.

- Then $S \cong \mathbb{RP}^2$.

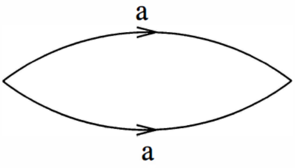

Figure 1.4. *The projective plane obtained from a bigon via identifications.*

Special case: Suppose P has two adjacent edges pointing in the same direction, with other edges present.

- The dotted arc separating these edges must cut a disc off the other side (otherwise S is not prime).
- Hence $S \cong \mathbb{RP}^2$.

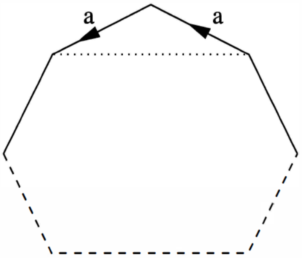

Figure 2.12. *A polygonal representation of S containing a projective plane.*

Assumption henceforth: No two adjacent edges of P are identified.

Step 2: Curves Arising from Arcs in P

Let c be a simple closed curve in S realized by an arc in P connecting a pair of identified edges.

We consider two cases based on whether c is **separating** or **nonseparating**. Suppose then that c is **separating**. Then c cuts S into two components S_1 and S_2.

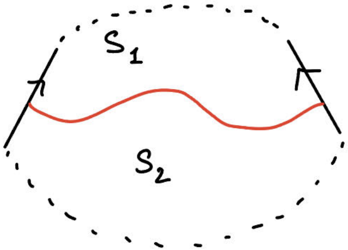

Figure: Regions S_1, S_2, induced by a separating curve c.

Step 2a: The Separating Case

Since S is **prime**, one component, say S_2, must be a **disc**.

- This disc S_2 meets remnants of edges of P.
- Therefore, there is a polygonal representation of S with **fewer edges**.

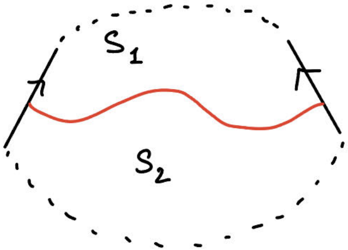

Figure: Regions S_1, S_2, induced by a separating curve c.

We replace P with this smaller polygon and continue.

Suppose c is **nonseparating**.

- Let a be a simple closed curve intersecting c **exactly once**.
- Let $C(c \cup a)$ be a **collar neighborhood** of $c \cup a$ in S.

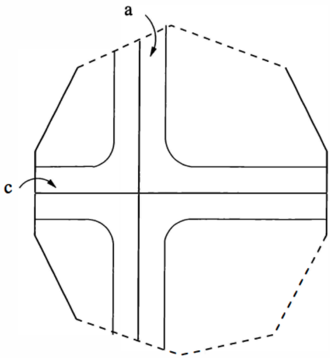

Figure 2.13. *The curve c and arc a.*

Figure

One can show that $C(c \cup a)$ is homeomorphic to one of:

1. a **torus minus a disc**,
2. a **Klein bottle minus a disc**, or
3. a **Möbius band minus a disc**.

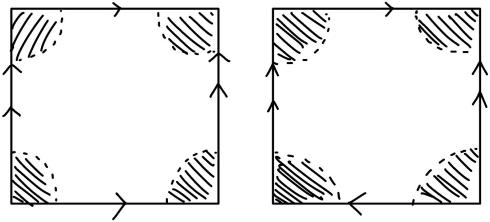

Figure: $\mathbb{T}^2$ minus a disk, $\mathbb{K}^2$ minus a disk

In each case, the boundary curve(s) of $C(c \cup a)$ must **bound discs** in S.

Step 2b: Conclusion of the Nonseparating Case

Recall the three cases for $C(c \cup a)$:

1. **Torus minus a disc:** The boundary bounds a disc in S, so $S \cong T^2$.
2. **Klein bottle minus a disc:** This would require $S \cong K$, but the Klein bottle is **not prime** (since $K \cong \mathbb{RP}^2 \# \mathbb{RP}^2$).
3. **Möbius band minus a disc:** The boundary bounds a disc in S, so $S \cong \mathbb{RP}^2$.

Note: The sphere is a prime surface, by Jordan Curve Theorem.
Conclusion: The only closed prime surfaces are S^2, T^2, and $\mathbb{RP}^2$.

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From Prime Surfaces to the Full Classification

Another way to look at it is... every compact surface decomposes as a **connected sum of prime surfaces**:

$$S \cong P_1 \# P_2 \# \cdots \# P_n$$

where each P_i is prime.

Since the only prime closed surfaces are S^2, T^2, and $\mathbb{RP}^2$:

- The decomposition of a surface into prime surfaces, as if $S = S_1 \# S_2$, then S_1 and S_2 admit polygonal representations whose polygons have fewer vertices than the one of S.

Lemma

$T^2 \# \mathbb{RP}^2 = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

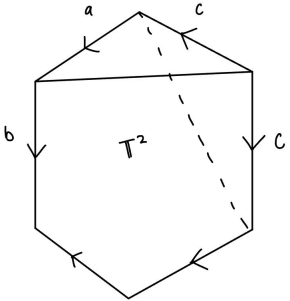
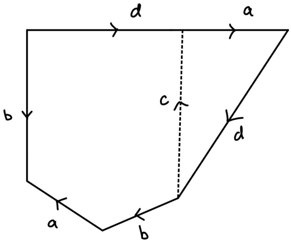
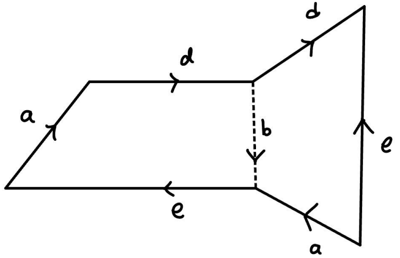


Figure: $\mathbb{T}^2 \# \mathbb{RP}^2$



Figure

From Prime Surfaces to the Full Classification



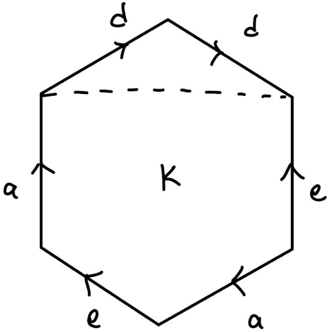


Figure: $\mathbb{K}^2 \# \mathbb{RP}^2$

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Summary

The classification of compact surfaces is complete.